Thermodynamic approach to entanglement in many-body systems

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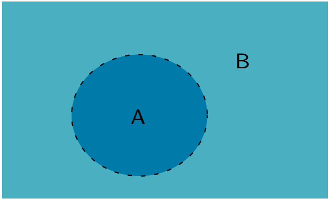
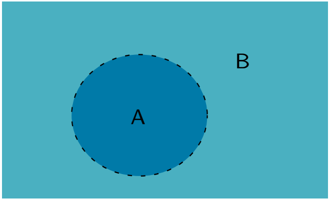
Measure of correlations



Measure of correlations

- Classical mutual information (distance from separable state) :

$$I_{cl}(A:B) \quad = \quad \sum_{a\in E_A, b\in E_B} p(a,b)\log\left(\frac{p(a,b)}{p(a)p(b)}\right)$$



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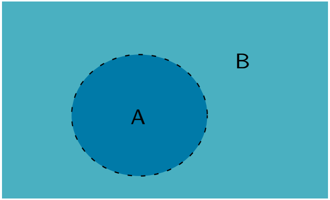

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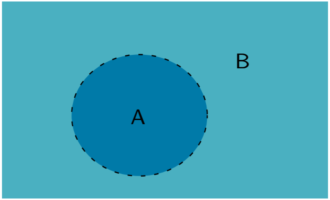
- Quantum mutual information :

$$I_{qu}(A:B) = S(\rho_A) + S(\rho_B) - S(\rho_{AB})$$

with $S(\rho) = -Tr(\rho \log \rho)$. Represents the number of classical bits of randomness we have to apply locally to erase all the correlations in the state ρ_{AB} (Groisman et al. PRA, 2005).



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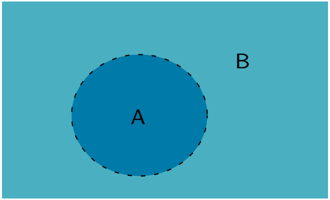

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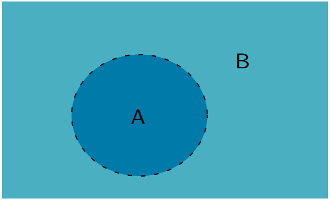

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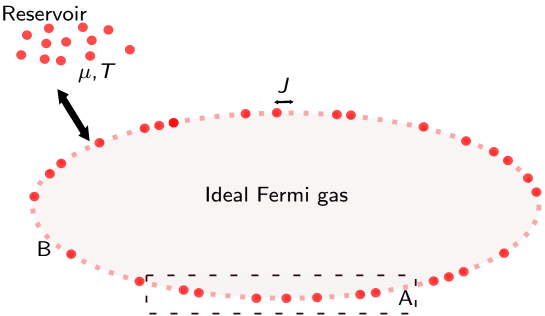
- How can we measure $I(A:B)$? Related to observable quantities ?

The model system under study : 1D Fermi gas of non-interacting particles



Hamiltonian :

$$\hat{H} = \sum_{i=1}^{\hat{N}} \frac{\vec{p_i}^2}{2m}$$

$$= \int_{R^d} d\vec{x} \hat{\Psi}^\dagger(\vec{x}) \left(-\frac{\hbar^2}{2m}\Delta\right) \hat{\Psi}(\vec{x})$$

$$=_{(1D\ \ lattice)} -J \sum_i \hat{c}_i^\dagger \hat{c}_{i+1} + \hat{c}_{i+1}^\dagger \hat{c}_i - 2\hat{c}_i^\dagger \hat{c}_i$$

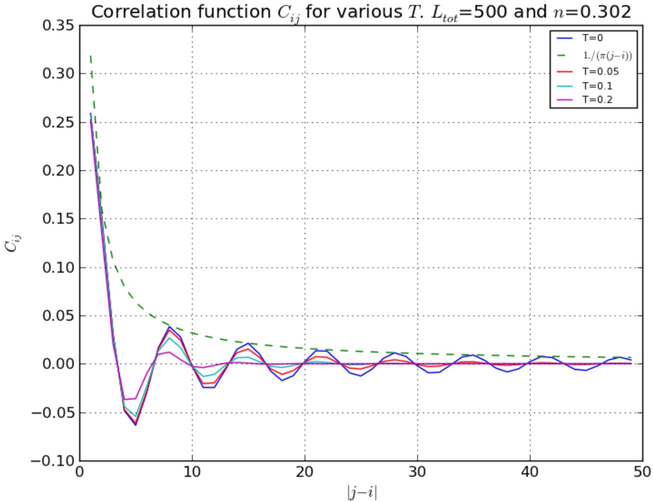
$$=_{(FT)} 2J \sum_{k=0}^{N-1} \left[1 - cos\left(\frac{2k\pi}{N}\right)\right] \hat{c}_k^\dagger \hat{c}_k$$

Density-matrix :

$$\rho_{AB}(T, \mu) \propto \exp - \frac{\hat{H} - \mu\hat{N}}{T}$$

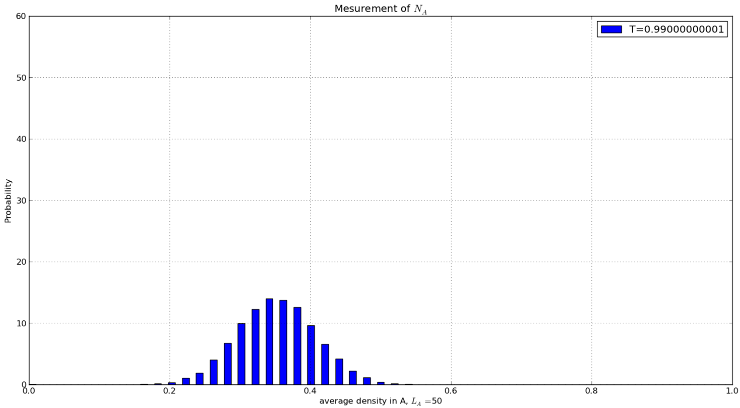
Correlations without interactions ?

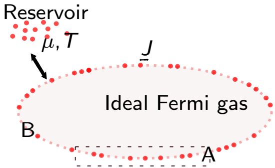
$$C_{ij} = \langle \hat{c}_i^\dagger \hat{c}_j \rangle$$



- At $T = 0$: $C_{ij} = \frac{\sin(\pi n(j-i))}{\pi(j-i)}$ (at $L_{tot} \to \infty$).
- At $T \neq 0$: $|C_{ij}| \sim \exp -\frac{|j-i|}{\xi}$ at large distance.
- These correlations (at any T) have a purely quantum origin

A 'numerical experiment'





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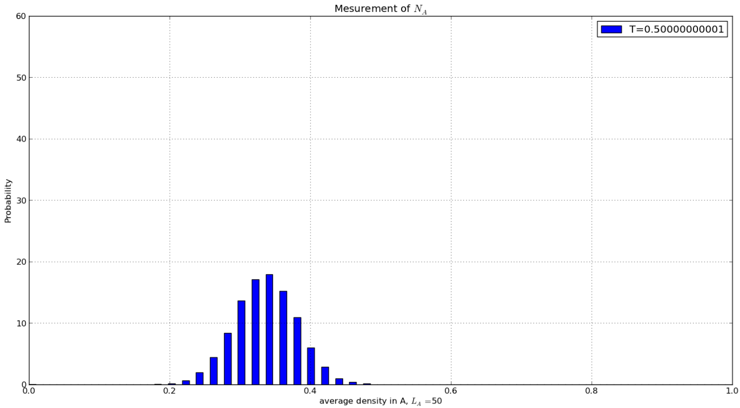

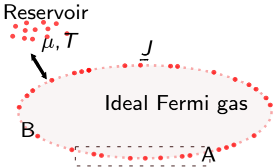

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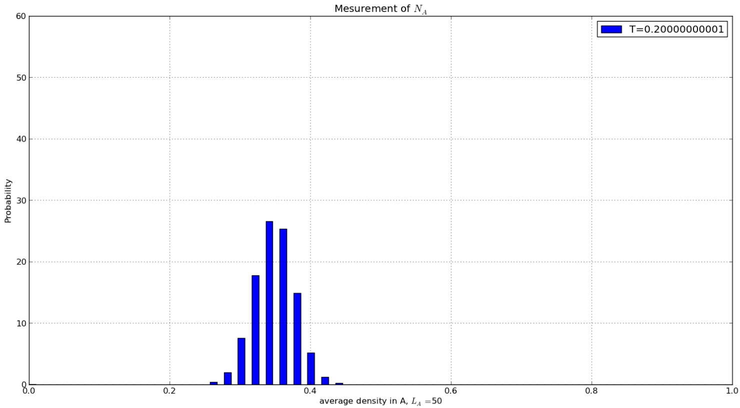

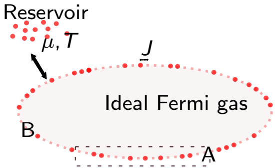

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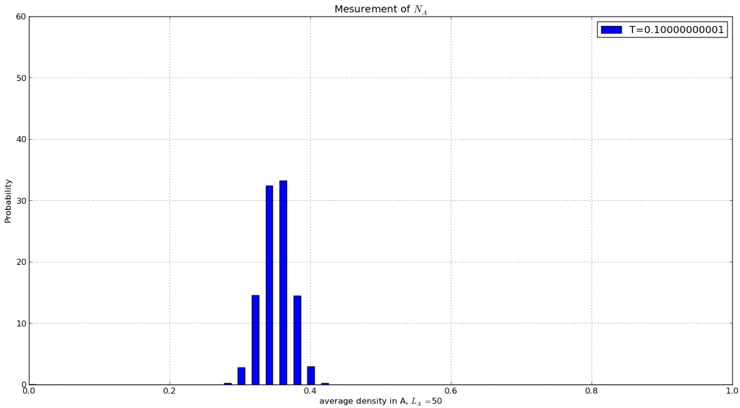

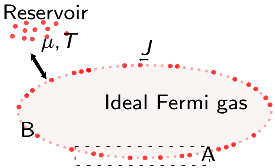

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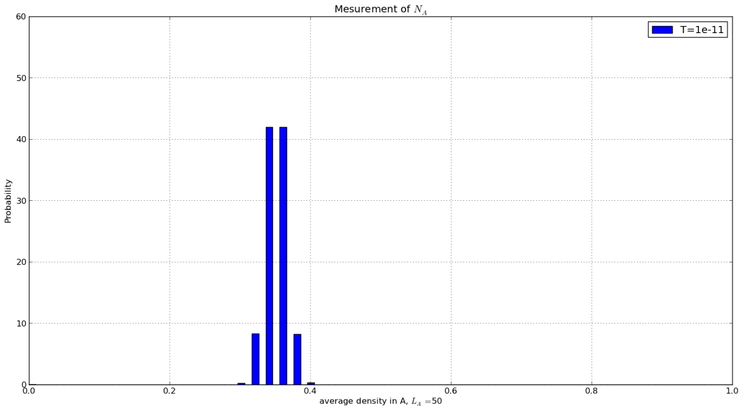
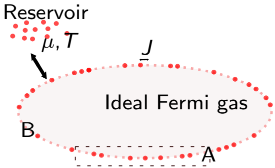
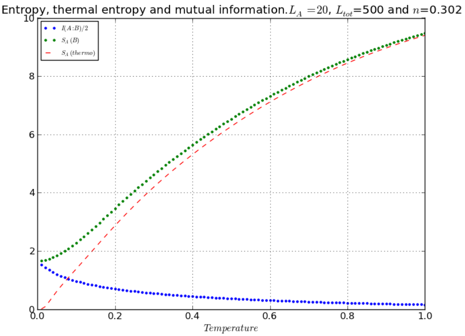


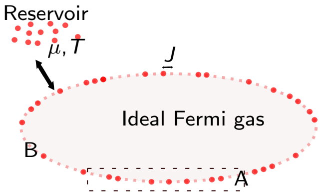


Entropy and correlations



Up to small boundary terms :

$$S_A(B) \approx S_A(thermo) + \frac{1}{2} I(A:B)$$



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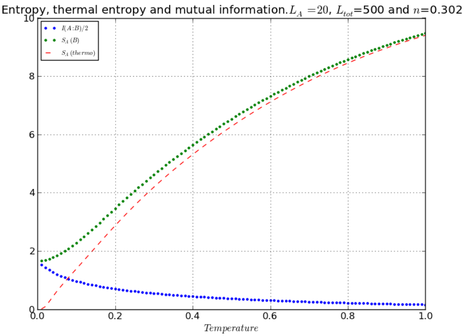

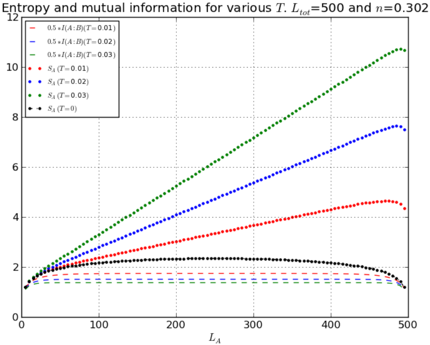

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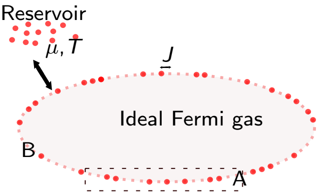

Two regimes (limit $1 \ll L_A \ll L_{tot}$):

- $T = 0$: $S_A \sim \frac{1}{3} \log(L_A)$
  Critical behaviour
- $T \neq 0$: $S_A \sim S_A(thermo)$
  $I(A:B) \to c(T)$: boundary law.

Towards an 'effective grand-canonical description' of quantum fluctuations

- Several works devoted to the relation between entanglement entropy and fluctuations (Song et al. PRB 2012), but no intuitive picture.
- Question : Zero-point fluctuations are an intuitive consequence of Heisenberg principle. Is entanglement entropy $S_A(T = 0)$ the 'information' counterpart of these fluctuations ?
- Motivation : It would be nice to capture the quantum effects into an 'entanglement pseudo-temperature' (cf. $\delta^2 N_A(B) > \delta^2 N_A(thermo)$).

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- Definition :

Definition through a thermodynamical fluctuation/response relation :

$$T_{ent} \equiv \frac{\langle \delta^2 N_A \rangle}{\frac{\partial \langle N_A \rangle}{\partial \mu_A}}$$

- Intuition :

$$\left\{ \begin{array}{c} \text{Sub-system } A \text{ of size } L_A \\ \text{immersed in the Fermi sea} \end{array} \right\} \Longleftrightarrow \left\{ \begin{array}{c} \text{same System of free Fermions} \\ \text{in the Grand-Canonical ensemble} \\ \text{with } T_{ent}(L_A) \end{array} \right\}$$

More about $T(\hat{N}_A) = \frac{\langle \delta^2 N_A \rangle}{\frac{\partial \langle N_A \rangle}{\partial \mu_A}}$.

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$$\hat{H}(\mu_A) = \hat{H}_0 - \mu_A \hat{N}_A$$

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- We define the *quantum variance* :

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Equality only occurs if $[\hat{N}_A, \hat{H}_0] = 0$.

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It can also be shown that :

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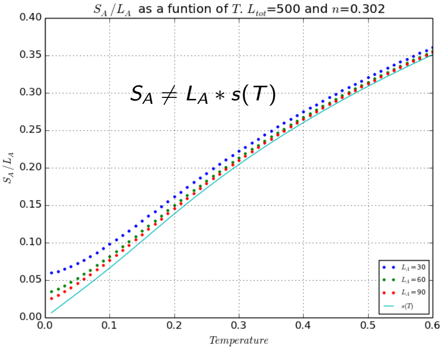
Relevance of $T(\hat{N}_A)$.



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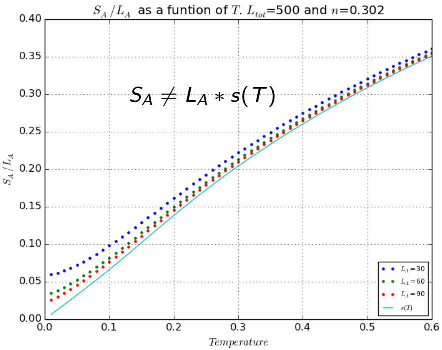
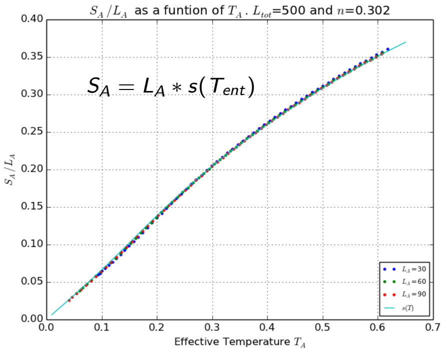
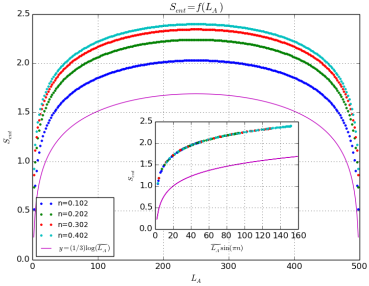




Entanglement Thermodynamics at $T = 0$

- $$S_A = \frac{1}{3} \ln L_A + O(1)$$

(Calabrese, Cardy, J.Stat.Mech, 2004)



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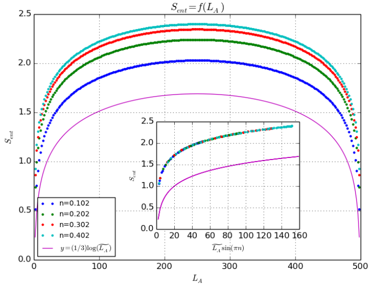

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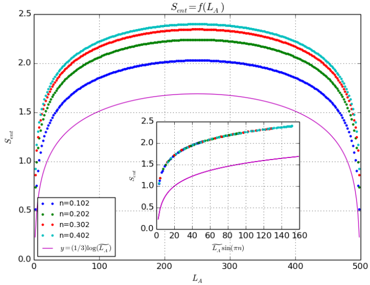

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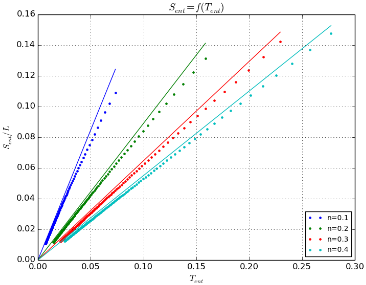

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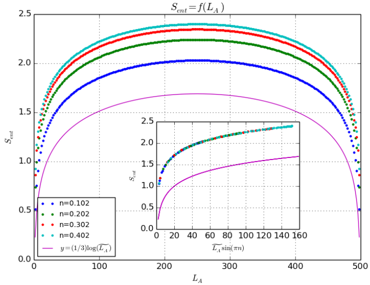

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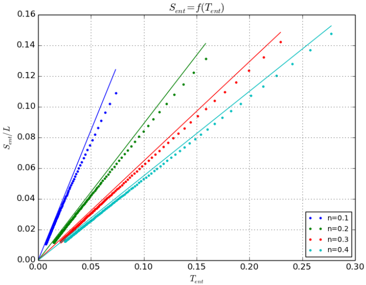

- Entanglement Temperature through E_A.

$$\begin{aligned} H_0 &\longrightarrow H_0 - t_A H_A \\ \frac{\partial \langle H_A \rangle}{\partial t_A} &= \mu^2 \rho(\mu) L_A + O(1) \\ \delta^2 E_A &= \left(\frac{\mu}{\pi}\right)^2 \ln L_A + O(1) \end{aligned}$$

$$T(E_A) = T(N_A) + O(1/L_A)$$

How to measure entanglement ?

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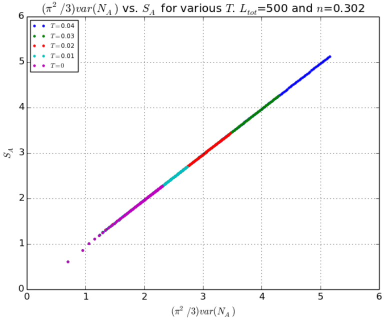

$$S_A(T, L_A) \approx \frac{\pi^2}{3} var(N_A(T, L_A))$$

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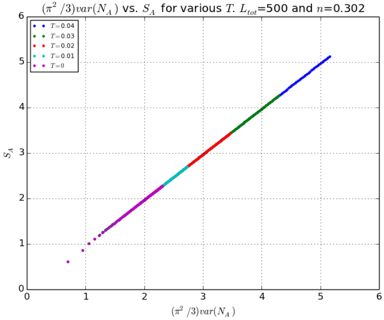

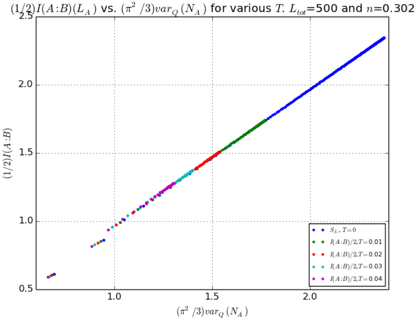

$S_A(T, L_A) \approx \frac{\pi^2}{3} var(N_A(T, L_A))$

$\frac{1}{2} I(A:B) \approx \frac{\pi^2}{3} var_Q(N_A)$

Outlook

- T_{ent} is a good parameter to gain intuition about the thermodynamics of a subsystem immersed in a ground-state.
- We are not saying that $\rho_A \sim e^{-\beta_{ent} H_A}$, which we know is wrong ! (cf. entanglement spectrum shape)
- The possibility to measure correlations through local measurements opens interesting perspectives.

What we plan to study :

- Extension to $d = 2$ and search for analytical results.
- Extension to weakly interacting bosons (same tools).
- Extension to systems with more interesting classical limit.

Thank you for your attention !